

GEOMETRIC PROGRESSION-FREE SEQUENCES WITH SMALL GAPS II

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ABSTRACT. When k is a constant at least 3, a sequence S of positive integers is called k -GP-free if it contains no nontrivial k -term geometric progressions. Beiglböck, Bergelson, Hindman and Strauss first studied the existence of a k -GP-free sequence with bounded gaps. In a previous paper the author gave a partial answer to this question by constructing a 6-GP-free sequence S with gaps of size $O(\exp(6 \log n / \log \log n))$. We generalize this problem to allow the gap function k to grow to infinity, and ask: for which pairs of functions (h, k) do there exist k -GP-free sequences with gaps of size $O(h)$? We show that whenever $(k(n) - 3) \log h(n) \log \log h(n) \geq 4 \log 2 \cdot \log n$ and h, k satisfy mild growth conditions, such a sequence exists.

1. INTRODUCTION

Let S be an increasing sequence of positive integers. We say that S is k -GP-free if it contains no k -term geometric progressions with common ratio not equal to 1, where $k \geq 3$ for the problem to be nontrivial. Let h be a nondecreasing function $\mathbb{N} \rightarrow \mathbb{R}^+$. We say that a sequence S has gaps of size $O(h)$ if there exists a constant $C > 0$ such that for every pair $m, N \in \mathbb{N}$ with $m \leq N$, the sequence S intersects the interval $[m, m + Ch(N))$.

The maximal asymptotic density of a k -GP-free sequence is well-studied [3, 10, 11, 15]. Beiglböck et al. [2] originally posed the related question:

Problem 1. Does there exist $k \geq 3$ and a k -GP-free sequence S such that S has gaps of size $O(1)$?

The standard example of a 3-GP-free sequence is the sequence Q of positive squarefree numbers $1, 2, 3, 5, 6, 7, 10, \dots$, which has asymptotic density $\frac{6}{\pi^2}$. Despite its large density, the size of its largest gaps is not known. The best unconditional result available is that of Filaseta and Trifonov [5] that Q has gaps of size $O(N^{1/5} \log N)$, and Trifonov also established a generalization that the sequence of k -th-power-free numbers has gaps of size $O(N^{1/(2k+1)} \log N)$ [16]. Assuming the *abc* conjecture, Granville showed that the gaps of Q are of size $O(N^\varepsilon)$ for all $\varepsilon > 0$ [7].

All of these bounds can be improved immensely if we assume the conjecture of Cramér that the gaps between consecutive primes are $O(\log^2 N)$ [4]. For a discussion of Cramér's model and implications, see the article of Pintz [12]. The problem of bounding largest gaps between consecutive primes, both from above and below, is notoriously difficult, and the best known lower bound is

$$p_{n+1} - p_n \geq \frac{C \log p_n \log \log p_n \log \log \log p_n}{\log \log \log p_n}$$

for some $C > 0$ and infinitely many n , due to Ford, Green, Konyagin, Maynard, and Tao [6], an improvement by $\log \log \log p_n$ over the longstanding bound of Rankin [14]. The best

unconditional upper bound is $p_{n+1} - p_n = O(N^{0.525})$, due to Baker, Harman, and Pintz [1], with $O(N^{1/2} \log N)$ possible assuming the Riemann hypothesis.

Instead of pursuing these notoriously difficult problems, in a previous paper the author showed that by replacing Q by a randomly constructed analogue, we can improve on Granville's bound unconditionally.

Theorem 2. [8] *There exists a 6-GP-free sequence T and a constant $C > 0$ such that the gaps of T are of size $O(\exp(C \log N / \log \log N))$. In fact C can be taken to be any positive real greater than $\frac{5}{6} \log 2$.*

In this paper we generalize the Problem 1 as follows. Henceforth k is no longer a constant but a nondecreasing function $k : \mathbb{N} \rightarrow \mathbb{R}_{\geq 3}$. We say that S is k -GP-free if for every $N \in \mathbb{N}$, the finite subsequence $S \cap \{1, 2, \dots, N\}$ does not contain any nontrivial geometric progressions of length at least $k(N)$.

Problem 3. For which pairs of functions (h, k) do there exist k -GP-free sequences S such that S has gaps of size $O(h)$?

We call h the gap function and k the length function, and a pair (h, k) feasible if such an S exists. Thus far we have only dealt with constant length function; in particular Theorem 2 shows that the pair $(\exp(C \log N / \log \log N), 6)$ is feasible. At the other end of the spectrum, it is trivial that $(1, \log N / \log 2)$ is a feasible pair, simply because the longest possible geometric progression in $1, \dots, N$ has length at most $\log N / \log 2$. In the last section of this paper we show in fact that $(1, \varepsilon \log N)$ is feasible for any $\varepsilon > 0$.

To interpolate between these two situations, we prove the following theorem, extending the method used in [8] to prove Theorem 2.

For two functions $f, g : \mathbb{N} \rightarrow \mathbb{R}^+$ we write $f = O(g)$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(n)$ for all $n \in \mathbb{N}$ and $f = o(g)$ if for every $C > 0$ the inequality $f(n) \leq Cg(n)$ holds for all n sufficiently large. We also write $f = \Omega(g)$ if $g = O(f)$.

Theorem 4. *Let (h, k) be nondecreasing functions $\mathbb{N} \rightarrow \mathbb{R}^+$ such that $h(n) = \Omega((\log x)^{1/(1-\log 2)})$ and for all sufficiently large n , $k(n) > 5$. If they satisfy*

$$(k(n) - 3) \log h(n) \log \log h(n) \geq 4 \log 2 \cdot \log n,$$

for all sufficiently large n , then there exists a k -GP-free sequence T with gaps of size $O(h)$.

As a corollary, if k is constant we recover Theorem 2 with a weaker constant.

2. PRELIMINARIES

In this section we generalize the GP-free process of [8] to probabilistically construct a k -GP-free sequence. First we simplify Theorem 4 by reducing the set of possible length functions k . It suffices to show the following.

Theorem 5. *If k is a nondecreasing function $\mathbb{N} \rightarrow \{6, 8, \dots\}$ taking on even positive integer values at least 6, and $h : \mathbb{N} \rightarrow \mathbb{R}^+$ is a function satisfying $h(n) = \Omega((\log x)^{1/(1-\log 2)})$, $h(n) = o(\sqrt{n})$ and*

$$(k(n) - 2) \log h(n) \log \log h(n) \geq 4 \log 2 \cdot \log n,$$

for all n sufficiently large, then there exists a k -GP-free sequence T with gaps of size $O(h)$.

Proof. (that Theorem 5 implies Theorem 4). Suppose Theorem 5 is true, and let k be as in Theorem 4. We can certainly round up k to the nearest integer to begin with. It is also possible to ignore the finite set of n for which $k \leq 5$, since we only care about n sufficiently large. If we round k down to the nearest even integer, if it originally satisfied the inequality of Theorem 4, then it has decreased by at most 1 uniformly, so the inequality above holds. Finally, if we prove the theorem for all $h(n) = o(\sqrt{n})$, then it follows for all larger h as well, so we may as well assume $h(n) = o(\sqrt{n})$. \square

Let G_k be the family of all geometric progressions of positive integers such that if t is the largest term, then the length is at least $k(t)$. Enumerate them as $G_{k,i}$ in order lexicographically as sequences of positive integers. We assume that each $G_{k,i}$ has common ratio $r_{k,i} > 1$.

Furthermore, there may be longer $G_{k,i}$ containing shorter ones; let G_k^* denote the result of removing from G_k all $G_{k,i}$ which contain some $G_{k,j}$ with $j \neq i$. Thus to find a k -GP-free sequence it suffices to construct a sequence T_k missing at least one element from each progression in G_k^* . Let $G_{k,i}^*$ denote the i -th progression in G_k^* .

Definition 6. For a nondecreasing function $k : \mathbb{N} \rightarrow \{6, 8, \dots\}$, define the k -GP-free process as follows. Define an integer-sequence valued random variable $U_k = (u_1, u_2, \dots)$ where $u_i \in G_{k,i}^*$ such that if

$$G_{k,i}^* = (a_i b_i^{k-1}, a_i b_i^{k-2} c_i, \dots, a_i c_i^{k-1}),$$

then u_i is chosen from $a_i b_i^{k/2-1} c_i^{k/2}$ and $a_i b_i^{k/2} c_i^{k/2-1}$ with equal probability $\frac{1}{2}$. Each u_i is picked independently of the others. Then T_k is the random variable whose value is the sequence of all positive integers never appearing in U_k , sorted in increasing order.

It is clear that T_k is k -GP-free by definition, as it misses at least one term out of each $G_{k,i}^*$. We now bound the probability that a given $n \in \mathbb{N}$ lies in T_k generated as above. For $i, j \geq 1$, let $d(n; i, j)$ count the number of ways to factor $n = ab^i c^j$ for some $a, b, c \in \mathbb{N}$.

Lemma 7. For a positive integer n , the sequence T_k constructed in Definition 6 contains n with probability

$$\mathbb{P}[T_k \ni n] \geq 2^{-d(n; k(m)/2, k(m)/2-1)},$$

where m is any positive integer such that any $G_{k,i}^*$ containing n in its middle two terms has largest term at least m .

Proof. The inequality is equivalent to the statement that n is one of the middle two terms in at most $d(n; \frac{k(m)}{2}, \frac{k(m)}{2} - 1)$ progressions of G_k^* . We form an injective correspondence from progression $G_{k,i}^*$ containing n in the middle two terms to factorizations of n as $n = ab^{k(m)/2} c^{k(m)/2-1}$. If a progression

$$G_{k,i}^* = (a_i b_i^{k'-1}, a_i b_i^{k'-2} c_i, \dots, a_i c_i^{k'-1})$$

with $b_i < c_i$ and $k' \geq k(a_i c_i^{k'-1})$ contains n as one of the middle two terms, then certainly $k(m) \leq k'$. Supposing $n = a_i b_i^{k'/2-1} c_i^{k'/2}$, we map $G_{k,i}^*$ to the factorization $n = ab^{k(m)/2} c^{k(m)/2-1}$ with $a = a_i b_i^{(k'-k(m))/2} c_i^{(k'-k(m))/2}$, $b = c_i$ and $c = b_i$. Similarly if $n = a_i b_i^{k'/2} c_i^{k'/2-1}$ we take $a = a_i b_i^{(k'-k(m))/2} c_i^{(k'-k(m))/2}$, $b = b_i$ and $c = c_i$. It is easy to see from the assumptions that $b_i < c_i$ and that no progression in G_k^* strictly contains another that the correspondence above is injective, as desired. \square

From here we can control the total probability that T_k misses an entire interval of the form $[x, x + Ch(x))$.

Lemma 8. *For a gap function $h(x) = o(x^{1-1/(k(x)-1)})$ and a constant $C > 0$, the sequence T_k constructed in Definition 6 satisfies $T_k \cap [x, x + Ch(x)) = \emptyset$ with probability*

$$\mathbb{P}[T_k \cap [x, x + Ch(x)) = \emptyset] \leq \exp \left(- \sum_{n \in [x, x + Ch(x))} \exp \left(- \log 2 \cdot d \left(n; \frac{k(x)}{2}, \frac{k(x)}{2} - 1 \right) \right) \right)$$

for all x sufficiently large.

Proof. We first prove that the events $\mathbb{P}[T_k \ni n]$ for $n \in [x, x + Ch(x))$ are mutually independent whenever x is sufficiently large. It suffices to show that no progression in G_k^* has both middle terms in the interval. Considering the difference between the two middle terms in a $G_{k,i}^*$, and assuming both lie inside $[x, x + Ch(x))$, we have

$$\begin{aligned} |a_i b_i^{k/2-1} c_i^{k/2} - a_i b_i^{k/2} c_i^{k/2-1}| &\geq a_i b_i^{k/2-1} c_i^{k/2-1} \\ &\geq x/b_i \\ &\geq x^{1-1/(k(m)-1)} \\ &\geq x^{1-1/(k(x)-1)} \end{aligned}$$

where $k \geq k(m)$ depends on the largest term $m = a_i c_i^{k-1} > x$. It follows that assuming $h(x) = o(x^{1-1/(k(x)-1)})$, for any $C > 0$ the middle two terms in any $G_{k,i}^*$ with largest term at most x are further apart than $Ch(x)$ for any x sufficiently large.

Thus the events corresponding to each n in the interval are mutually independent, and we can bound the probability involved by a product

$$\mathbb{P}[T_k \cap [x, x + Ch(x)) = \emptyset] \leq \prod_{n \in [x, x + Ch(x))} \left(1 - 2^{-d(n; k(m)/2, k(m)/2-1)} \right),$$

by Lemma 7. Since the inequality $1 - t \leq e^{-t}$ holds for all real t we arrive at the bound

$$\mathbb{P}[T_k \cap [x, x + Ch(x)) = \emptyset] \leq \exp \left(- \sum_{n \in [x, x + Ch(x))} \exp \left(- \log 2 \cdot d \left(n; \frac{k(m)}{2}, \frac{k(m)}{2} - 1 \right) \right) \right).$$

Here each $m = m(n)$ can certainly be chosen as any number at most n . Thus we replace them all by x , arriving at the desired bound. \square

Note that since we assumed $h(x) = o(\sqrt{x})$ the growth condition in Lemma 8 is automatically satisfied.

3. PROOF OF THE MAIN THEOREM

All that remains is to give lower bounds for the sum

$$S(x, h, k, C) = \sum_{n \in [x, x + Ch)} \exp \left(- \log 2 \cdot d \left(n; \frac{k}{2}, \frac{k}{2} - 1 \right) \right),$$

where $k = k(x)$ and $h = h(x)$ are functions satisfying the conditions of Theorem 5. To this end we break down $[x, x + Ch)$ into two sets, one of which has few $(k/2 - 1)$ -power divisors, and restrict the sum to that set.

Lemma 9. *There is a positive constant B independent of x such that for all sufficiently large x ,*

$$S(x, h, k, C) \geq BC h(x) \exp \left(-\log 2 \exp \left(\frac{4 \log 2 \cdot \log x}{(k(x) - 2) \log h(x)} \right) \right).$$

Proof. Fix an $x > 0$ and write $k = k(x)$, $h = h(x)$. Denote by A the subset of $[x, x + Ch)$ consisting of all n divisible by some $p^{k/2-1}$, where $p \leq h$. We can bound the size of A by

$$\begin{aligned} |A| &\leq \sum_{\text{prime } p \leq h} \left(\frac{Ch}{p^{k/2-1}} + 1 \right) \\ &\leq (\zeta(k/2 - 1) - 1)Ch + o(h), \end{aligned}$$

where ζ is the Riemann zeta function and we used the elementary Chebyshev bound $\pi(h) = o(h)$ on the prime-counting function π . Since $k \geq 6$ and $\zeta(t) - 1 < 1$ uniformly on $t \geq 2$, there exists a constant B such that for x , and thus h , sufficiently large, $|A| \leq (1 - B)Ch$.

If $n \notin A$, we can factor $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} n'$ where n' is $(k/2 - 1)$ -th power free, each $\alpha_i \geq k/2 - 1$, and each $p_i \geq h$ is prime. As a result,

$$\sum_i \alpha_i \leq \frac{\log n}{\log h},$$

so by a smoothing argument we can bound $d(n; \frac{k}{2}, \frac{k}{2} - 1)$ subject to these assumptions,

$$d\left(n; \frac{k}{2}, \frac{k}{2} - 1\right) \leq \exp \left(\log 2 \cdot \frac{\log n}{(k/2 - 1) \log h} + \log 2 \cdot \frac{\log n}{(k/2) \log h} \right),$$

where we simply bounded the number of pairs b, c satisfying $b^{k/2-1} | n$ and $c^{k/2} | n$. Summing up over all terms in $[x, x + Ch)$ outside A , we get

$$S(x, h, k, C) \geq BC h \exp \left(-\log 2 \exp \left(\left(\frac{1}{k} + \frac{1}{k-2} \right) \frac{(2 \log 2) \cdot \log x}{\log h} \right) \right),$$

and finally replacing $1/k \leq 1/(k-2)$ we have the desired inequality. \square

Finally, we prove Theorem 5 using Lemma 9.

Proof. (of Theorem 5). By Lemma 8 it suffices to pick h, k such that the sum of probabilities

$$\sum_{x \geq 1} \mathbb{P}[T_k \cap [x, x + Ch(x)) = \emptyset] \leq \sum_{x \geq 1} \exp(-S(x, h, k, C)) < 1$$

for C sufficiently large, forcing the probability of finding a T with gaps $O(h)$ to be nonzero. This will hold as long as the sum converges for some fixed C ; making C large enough will make the sum arbitrarily small. Now, suppose that $(k-2) \log h \log \log h \geq 4 \log 2 \cdot \log n$ as in Theorem 5. Then, applying the inequality of Lemma 9, we have

$$\begin{aligned} S(x, h, k, C) &\geq BC h \exp(-\log 2 \log h) \\ &\geq BC h^{1-\log 2}, \end{aligned}$$

and finally since $h = \Omega((\log x)^{1/(1-\log 2)})$, we get

$$\sum_{x \geq 1} \exp(-S(x, h, k, C)) \leq \sum_{x \geq 1} x^{-BCD},$$

for some constant $D > 0$, so picking C for which $BC > 1$ gives a convergent sum. \square

4. CLOSING REMARKS

The goal of this paper was to interpolate smoothly between the two feasible pairs $(h, k) = (\exp(C \log N / \log \log N), 6)$ and $(h, k) = (1, \log N / \log 2)$, and we recover both pairs, up to constants, in the relation

$$(k(n) - 3) \log h(n) \log \log h(n) \geq 4 \log 2 \cdot \log n.$$

Unfortunately, when k is sufficiently close to $\log n$, then the method of Theorem 4 fails because $h = o((\log x)^{1/(1-\log 2)})$. Nevertheless, we expect all pairs (h, k) which satisfy this inequality to be feasible. In the case that $h = 1$ we can make an improvement on $(1, \log N / \log 2)$.

Proposition 10. *For any $\varepsilon > 0$, if $k(n) = \varepsilon \log n$ then there exists a k -GP-free sequence T with gaps of size $O(1)$.*

Proof. We say a positive integer m is divisible by a k -th power if $p^{\lceil k(m) \rceil} | m$ for some prime p , and that m is k -free otherwise. Consider the sequence T of all k -free integers; we claim that its gaps are uniformly bounded. In fact, note that if $p^{\lceil k(m) \rceil} | m$ then

$$\begin{aligned} p^{k(m)} &\leq m \\ \varepsilon \log m \cdot \log p &\leq \log m \\ \log p &\leq \frac{1}{\varepsilon}, \end{aligned}$$

and so p lies in the finite set of all primes less than $e^{1/\varepsilon}$. In particular, for x sufficiently large, the interval $[x, x + e^{1/\varepsilon} + 1)$ will contain at least one k -free number. Indeed, it is easy to check that each $p \leq e^{1/\varepsilon}$ contributes at most one multiple of $p^{k(x)}$ to that interval. \square

Further improvement in the case of h small or constant along these lines is blocked by the Chinese Remainder Theorem. In particular, for $k = o(\log n)$ and any constant h we can find infinitely many intervals $[x, x + h)$ in which each positive integer in $[x, x + h)$ is divisible by arbitrarily many $k(x)$ -th powers of primes.

The probabilistic method in Definition 6 is by no means optimal, but is defined in such a way to guarantee the independence of events in an interval $[n, n + Ch)$. We expect that a sophisticated study of redundancies in our method can substantially improve at least the constant in Theorem 4.

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